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A linear differential game on a plane with a minimum-type functional☆

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Abstract

A differential game on a plane with a functional in the form of the minimum, with respect to time, of a certain prescribed phase vector function (quality function) is considered. It is proved that the game value is constant outside a certain bounded region, consisting of two parts. In the first subregion, the value is equal to the quality function, and in the second it satisfies Bellman's equation. For the constant-value region, where the players' optimum strategies are not unique, single-valued guaranteeing players' strategies are proposed. The results of a numerical investigation of the problem are presented.
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Problems of the theory of optimal control and differential games with a functional in the form of the minimum of a certain phase vector function along a trajectory of a dynamical system arise in applications and are of interest from a theoretical point of view. In a problem of this kind, Bellman's so-called optimality principle does not apply. Dynamic programming equations are only satisfied in a subregion of phase space, bounded by an unknown surface. In the remaining part of space, called the primary region, the function of the optimal result (the game value) is equal to the phase vector function determining the functional. Thus, an element of the solution of the problem is finding the unknown (free) boundary.

Below, we consider the case where the level lines of the mentioned phase variable function (the quality function) are ellipses. It is shown that the game value is equal to a constant quantity on the entire plane, with the exception of a certain bounded set that also includes the primary region. An algorithm is proposed for constructing the game value in the remaining part of game space. Since in the constant-value region of the game the optimal controls of the players are determined in a non-unique way, here single-valued guaranteeing player strategies are proposed that, in a certain subregion, solve the secondary problem of pursuit.

Problems of this kind arise in certain models describing the game counteraction of two aircraft or the control of aircraft with the aim of avoiding collision.

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1. Formulation of the problem

We will consider the game problem in a semi-infinite time interval with dynamics, initial data and constraints of the form

$$\begin{aligned} &\dot{x}_1 = x_2 + v, \quad \dot{x}_2 = u, \quad t \geq 0 \\ &x_1(0) = x_1^0, \quad x_2(0) = x_2^0, \quad |u| \leq 1, \quad |v| \leq 1 \end{aligned} \tag{1.1}$$

A certain scalar function $S(x)$ is specified, defined over the entire plane $x = (x_1, x_2) \in R^2$ and called the quality function. The pay-off in the game is the value of the minimum:

$$J[x^0, u, v] = \min_{t \geq 0} S(x(t; x_0, u, v)) \to \min_u \max_v, \quad x = (x_1, x_2) \tag{1.2}$$

where $x(t; x_0, u, v)$ is the solution of the equations of dynamics (1.1) in the semi-infinite interval $t \geq 0$, corresponding to the initial value x_0 and to certain admissible controls u and v of players P and E. Fig. 1 shows one of the solutions (trajectories) of system (1.1) and the level lines of the quality function. The minimum (1.2) corresponds to the point A in Fig. 1. Player P tends to minimize this quantity by selecting the control u, while player E tends to maximize it by selecting v.

It will be assumed that the quality function $S(x_1, x_2)$ is quadratic in its variables and has the form

$$S(x_1, x_2) = [x_1^2 - \varepsilon^2(x_1\cos\alpha + x_2\sin\alpha)^2 + x_2^2]^{1/2}, \quad 0 \leq \varepsilon \leq 1, \quad 0 \leq \alpha \leq \pi \tag{1.3}$$

where ε and α are parameters. Thus, the level lines of the function $S(x_1, x_2)$ are ellipses (circles when $\varepsilon = 0$) with their centres at the origin of coordinates; here, ε is the eccentricity of the ellipse and α is the angle of inclination to the abscissa axis of the major axis of the ellipse.

In accordance with a well-known procedure,[1,2] the differential game will be considered as a combination of minimax and maximin games for a guaranteed result for players P and E respectively. In the minimax game, the admissible controls of player P are considered to be positional controls (feedback controls), while for player E it is sufficient to consider the class of program controls. For the maximin game it is necessary to reset the classes of admissible controls. This implies formalization of the game by the procedure developed by Krasovskii and Subbotin.[1]

The quality function $S(x)$ has a global minimum equal to zero. This is reached at the origin of coordinates, at $x = 0$. If the dynamics (1.1) enabled player P to bring the phase vector from a certain set of initial states to the origin of coordinates, the game value on this set would be equal to zero. However, player P may bring the phase vector only to a certain level line (non-zero) of the function $S(x)$. In order to demonstrate this, an auxiliary quality game can be examined.

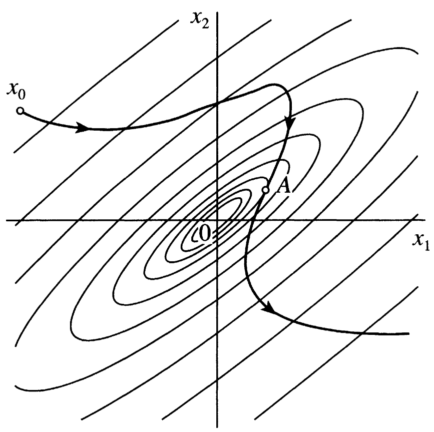

Fig. 1.

Differential games with a minimum-type functional in a limited time interval with the use of Subbotin's generalized optimality conditions[2] and numerical algorithms[3] were examined earlier in Ref. 4. A similar problem with simple motions in a semi-infinite time interval was considered in Ref. 5, singular characteristics being used to construct the unknown boundary.[6] Problems of control with a free boundary were studied in Ref. 7.

2. The auxiliary quality problem

We will consider the case where the level lines of the quality function $S(x)$ are circles, while the function itself has the meaning of the distance to the origin of coordinates:

$$S(x) = \sqrt{x_1^2 + x_2^2}$$

We will fix a circle of certain radius ρ and consider it to be the boundary of the terminal set C. In the auxiliary game, player P tends to bring the phase vector to set C, while player E tends to counteract this.

According to a well-known procedure,[8] it is first necessary to find the admissible region, which in the present case satisfies the inequality

$$\min_u \max_v (\nu_1(x_2 + v) + \nu_2 u) = \nu_1 x_2 + |\nu_1| - |\nu_2| \le 0$$

where $\nu = (\nu_1, \nu_2) = (x_1, x_2)$ is the vector of the normal to the terminal set at the point (x_1, x_2).

From this it follows that the admissible region consists of two arcs of a circle that lie between the half-branches Γ_1, ..., Γ_4 of four different hyperbolae, defined by the equation

$$x_1 x_2 + |x_1| - |x_2| = 0$$

$$\Gamma_1: x_2 = x_1/(1 - x_1), \quad 0 \le x_1 < 1; \quad \Gamma_3: x_2 = x_1/(1 + x_1), \quad -1 < x_1 \le 0 \tag{2.1}$$

$$\Gamma_2: x_2 = -x_1/(1 - x_1), \quad x_1 \le 0; \quad \Gamma_4: x_2 = -x_1/(1 + x_1), \quad x_1 \ge 0$$

The half-branches begin at the origin of coordinates; their corresponding fragments are shown in Fig. 2.

From the points A, B, A^* and B^* of intersection of these half-branches with the circle, it is necessary to run curves called barriers.[8] These are regular (classical) characteristics of the equation

$$\nu_1 x_2 + |\nu_1| - |\nu_2| = 0$$

(in which ν_1 and ν_2 are understood to be components of the gradient of a certain unknown function) and obey the system of ordinary equations

$$x_1' = -x_2 - 1, \quad x_2' = 1, \quad \nu_1' = 0, \quad \nu_2' = \nu_1 \quad \text{for barrier } B_1$$

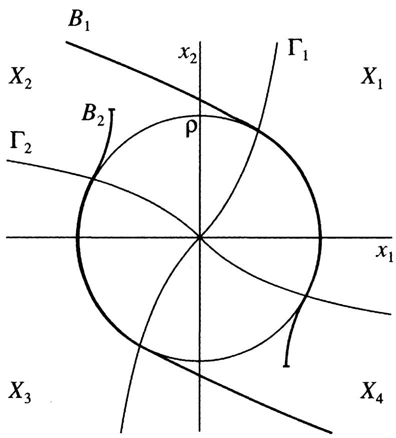

Fig. 2.

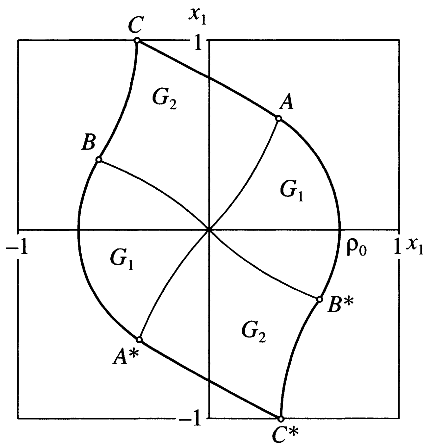

Fig. 3.

$$x_1' = -x_2 + 1, \quad x_2' = 1, \quad \nu_1' = 0, \quad \nu_2' = \nu_1 \quad \text{for barrier } B_2$$

The prime denotes differentiation with respect to the inverse time $\tau = T - t$. The barriers B_1 and B_2 are shown in Fig. 2.

We will show that barrier B_1 departs to infinity, while barrier B_2 ends at $x_2 = 1$. For this there is no need to examine the complete solution of these systems. It is sufficient to ascertain that the system for barrier B_k ($k = 1, 2$) has a first integral of the form

$$x_1 + (-1)^k (x_2 + (-1)^k)^2/2 = C_k, \quad k = 1, 2$$

The quantities ν_1 and ν_2, as normals to the barrier, are components of the gradient of the given first integrals. These components for barrier B_1 do not change sign, while for barrier B_2 they change sign at $x_2 = 1$. According to a well-known procedure,[8] this denotes the departure to infinity of barrier B_1 and the end of barrier B_2 (Fig. 2).

The greatest value of ρ at which the barriers intersect and provide division of game space is found from the system of algebraic equations

$$\begin{aligned} &x + (y+1)^2/2 = \tilde{x} + (\tilde{y} - 1)^2/2 + 2 \\ &xy + x - y = 0, \quad \tilde{x}\tilde{y} - \tilde{x} - \tilde{y} = 0 \\ &x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2 = \rho^2 \end{aligned} \tag{2.2}$$

and is equal to $\rho_0 \approx 0.69$. Here, (x, y) and $(\tilde{x}, \tilde{y})$ are the coordinates of points A and B. The barriers separate out region G (Fig. 3) bounded by sections of the barriers and arcs of a circle of radius ρ_0.

3. The game value in the initial problem

On constructing barriers, player E is able not to miss the phase point inside region G. On the other hand, as follows from the results of the next section, player P may bring the phase vector from any initial point outside region G to the boundary of this set (possibly, in infinite time). This means that, outside region G, the game value (1.1), (1.2) is constant and equal to ρ_0.

Region G is divided into two subsets G_1 and G_2 such that, at points of the set G_1, the minimum (1.2) is reached at the initial instant of time, and thereby the game value is equal to

$$V(x) = S(x), \quad x \in G_1$$

while at points of the set G_2 the minimum is reached at a future instant of time. Here, the function $S(x(t))$ has a minimum point with respect to t, for which the equality

$$\min_u \max_\upsilon \dot{S}(x_1, x_2) = \min_u \max_\upsilon (S_{x_1}(x_2 + \upsilon) + S_{x_2} u) = \frac{x_1 x_2 + |x_1| - |x_2|}{\sqrt{x_1^2 + x_2^2}} = 0$$

is satisfied, i.e. the minimum points are positioned on the hyperbolae (2.1).

These hyperbolae divide the entire game plane X into four quadrants $X_1, \ldots, X_4$ with curved boundaries $\Gamma_1, \ldots, \Gamma_4$ (Fig. 2). The regions indicated correspond to a certain sign of the time derivative of the quality function:

$$\dot{S}(x_1, x_2) > 0 \quad (x_1, x_2) \in X_1 + X_3; \quad \dot{S}(x_1, x_2) < 0 \ (x_1, x_2) \in X_2 + X_4$$

The subregion G_1 consists of two sectors of a circle C_0 (of radius ρ_0), obtained by the intersection of C_0 with the region $X_1 + X_3$. Here, the derivative $\dot{S}$ is positive, and player E achieves his/her best result – retention of the initial value of the quality function $S_0(x_0)$.

In region G_2, player P manages to reduce the initial value of the quality function. To construct the game value here it is necessary to examine one other secondary game of bringing the phase point to one of the curves Γ_i with terminal functional $S(x)$.

The constructions will be carried out for a wider game space $X_2 + X_4$, and we will then adopt a limitation of the game value to the region G_2.

To fix our ideas, we will consider, the subregion $G_2 \cap X_2$; constructions in $G_2 \cap X_4$ are similar. Player P attempts to bring the phase vector from the initial point in region X_2 to one of the curves Γ_1 or Γ_2 and to obtain the minimum value of the function $S(x)$.

The solution of this game can be constructed using the method of classical characteristics. We will write the Bellman–Isaacs equation

$$\min_u \max_\upsilon (p_1(x_2 + \upsilon) + p_2 u) = 0 \tag{3.1}$$

where p_1, p_2 represent the gradient of the value function $V(x)$. The extremal controls $\bar{u}$ and $\bar{\upsilon}$, which provide the minimum and maximum (3.1), are defined by the expressions

$$\bar{u} = -\operatorname{sign} p_2, \quad \bar{\upsilon} = \operatorname{sign} p_1$$

The characteristic system is written in the following way

$$x_1' = -x_2 - \bar{\upsilon}, \quad x_2' = -\bar{u}, \quad p_1' = 0, \quad p_2' = p_1$$

It can be assumed that p_2 is positive for trajectories beginning at each of the hyperbolae Γ_1 and Γ_2 and entering region X_2, and that p_1 for $|x_2| < 1$ is positive when starting from Γ_1 and negative when starting from Γ_2. Consequently, the optimum trajectories, running from Γ_1 and Γ_2, intersect at points of a certain υ-dispersal curve (in the terminology of Ref. 8) (Fig. 4). We emphasize that, in the game considered, the υ-dispersal curve ends at $|x_2| = 1$; then the switching curve begins, but it is not used in constructions, as it lies outside region G_2.

For trajectories starting from the hyperbola Γ_k ($k = 1, 2$), the first integral exists

$$x_1 + (x_2 + (-1)^{k+1})^2/2 = C_k, \quad k = 1, 2$$

Suppose from a certain internal point (x_1, x_2) of the region $G_2 \cap X_2$ the optimum trajectory arrives at the point (x_1, x_2) of the terminal curve Γ_1. Then the coordinates of the points are connected by the relations

$$x_1 + (x_2 + 1)^2/2 = \bar{x}_1 + (\bar{x}_2 + 1)^2/2, \quad \bar{x}_1 \bar{x}_2 + \bar{x}_1 - \bar{x}_2 = 0$$

from which it is possible to find the relation

$$\bar{x}_1 = \bar{x}_1(x_1, x_2), \quad \bar{x}_2 = \bar{x}_2(x_1, x_2)$$

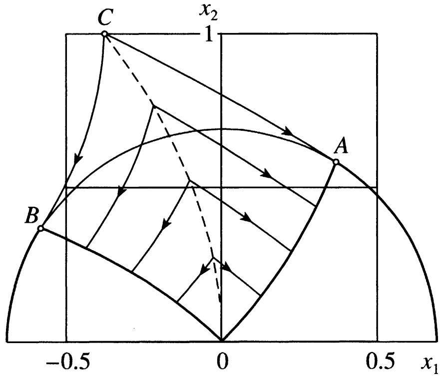

Fig. 4.

Then the corresponding branch of the game value is defined by the equation

$$V_1(x_1, x_2) = S(\bar{x}_1(x_1, x_2), \bar{x}_2(x_1, x_2))$$

The branch $V_2(x_1, x_2)$ is constructed similarly. We will define the functions

$$Q_2(x_1, x_2) = \max[V_1(x_1, x_2), V_2(x_1, x_2)], \quad Q_4(x_1, x_2) = \max[V_3(x_1, x_2), V_4(x_1, x_2)]$$

The functions $V_3(x_1, x_2)$ and $V_4(x_1, x_2)$ comprise smooth branches of the game value with initial points in $G_2 \cap X_4$.

The dispersal lines are found from the equations

$$V_1(x_1, x_2) = V_2(x_1, x_2), \quad V_3(x_1, x_2) = V_4(x_1, x_2)$$

From this it is possible to obtain the following parametric representation of the dispersal curve in the region $G_2 \cap X_2$:

$$x_2 = (s+1)^3(1-s)/(4s^2), \quad x_1 = 1 + (1-2s^2)/(2s^2) - (x_2+1)^2/2, \quad s_0 \le s \le 1$$

The origin of coordinates corresponds to $s = 1$, and the point at the level $x_2 = 1$ corresponds to the value $s = s_0 \approx 0.63$.

We will introduce into consideration the function

$$Q(x_1, x_2) = \begin{cases} Q_i(x_1, x_2), & (x_1, x_2) \in X_i, \quad i = 2, 4 \\ S(x_1, x_2), & (x_1, x_2) \in X_i, \quad i = 1, 3 \end{cases}$$

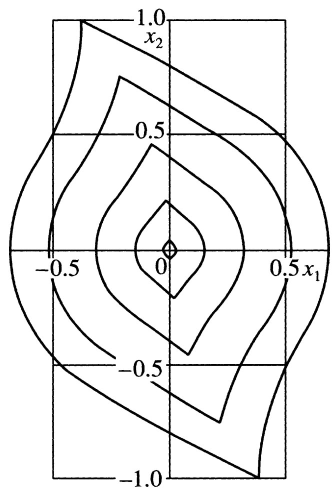

Fig. 5.

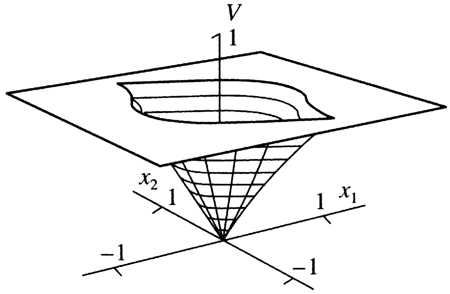

Fig. 6.

This function is defined and continuous in the region G_2. It can be shown that it is continuously differentiable on the lines Γ_i, i.e. the matching of the functions $Q_i(x_1, x_2)$ and $S(x_1, x_2)$ occurs smoothly. The function $Q(x_1, x_2)$ undergoes a gradient discontinuity only on dispersal curves. The level lines of this function are shown in Fig. 5.

Combining the solutions in the regions $G_2 \cap X_2$ and $G_2 \cap X_4$, it is possible to propose the following formula for the game value in the initial problem

$$V(x_1, x_2) = \begin{cases} Q(x_1, x_2), & (x_1, x_2) \in G \\ \rho_0, & (x_1, x_2) \in X_0 \quad (X_0 = X \backslash G) \end{cases} \tag{3.2}$$

The result is shown in Fig. 6 as a graph of the value function $V(x_1, x_2)$ as a function of the coordinates (x_1, x_2).

4. Guaranteeing strategies in the region X_0

If the initial game (1.1), (1.2) starts from the exterior X_0 of region G, then player E can behave passively right up until the boundary of region G is met. Player P must bring the phase vector to the boundary of set G and secure his/her result ρ_0, not being limited with respect to time. Thus, the optimum behaviour of both players is considerably ambiguous at X_0.

One possible mode of behaviour of both payers is to use optimum strategies for the secondary game of bringing the phase vector from region X_0 to the boundary of the δ-vicinity G_δ of set G, in which the pay-off is time. Player P attempts to minimize this time, while player E counteracts him/her. In constructions of this kind, it is convenient to use the δ-vicinity, rather than the set G itself, since player E can make the time at which G is met infinite.

Examination of the game of rapid response with the target set G_δ for an arbitrarily small $\delta > 0$ suggests the possible structure of players' strategies that is presented in Figs. 7 and 8. The phase portrait of trajectories that corresponds to the given strategies is shown in Fig. 9; meeting the set G by these trajectories requires, generally speaking, an infinite time.

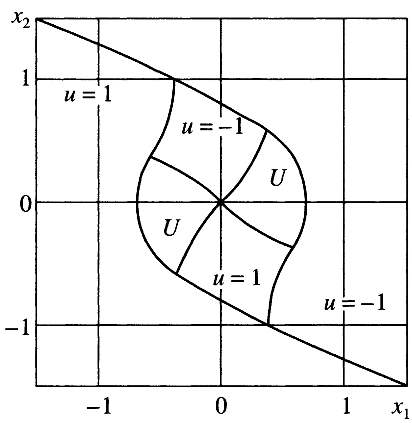

Fig. 7.

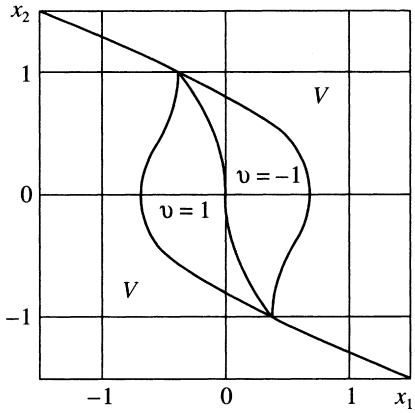

Fig. 8.

We will show that the strategies presented in Figs. 7 and 8 (which can be considered to be prescribed irrespective of the secondary game) guarantee players P and E, with the game starting from points of the set X_0, a value of ρ_0 of the quality functional with an arbitrary accuracy $\delta > 0$. Such a guarantee for player E means that, in the region X_0, he/she can use an arbitrary control but can keep the phase point from entering region G. The latter property of the strategy (Fig. 8) will be demonstrated in the next section.

Let us therefore consider the game from the viewpoint of player P. It can be shown that the strategy presented in Fig. 7 enables player P to bring (in the quickest way) the phase point from the upper (lower) half of the phase plane to the right-hand (left-hand) branch of barrier B_1 (see Fig. 9). Further, it is necessary to show that a constructive motion corresponding to the strategy in Fig. 7 in the sense of the known formalization[2] brings the phase point within a finite time to the vicinity of region G. Note that the strategies shown in Figs. 7 and 8 are piecewise-constant and are determined by the switching lines. A general approach to constructing and investigating controls by feedback using switching surfaces in linear differential games has been developed in Ref. 9.

We fix a certain number $\delta > 0$ and consider the δ-vicinity B_1^δ of the barrier line B_1

$$B_1^\delta = \{(x_1, x_2) : \left|x_1 + (x_2 + 1)^2/2 - \bar{c}\right| \leq \delta\} \ (\bar{c} = \bar{x}_1 + (\bar{x}_2 + 1)^2/2)$$

where (x_1, x_2) are the coordinates of a certain point lying on the barrier B_1, for example, the point A defined by system (2.2) (also see Fig. 3).

Player P uses a constant control +1 or −1 (depending on the region in which the phase point lies) in a time interval of length $h > 0$. We will determine the requirements that must be satisfied by the constant h, depending on the value of δ, with which the phase point during a constructive motion remains in the vicinity of B_1.

To fix our ideas, we will examine the left-hand branch of barrier B_1. Let the initial point $x_0 = (x_{10}, x_{20})$ lie below the barrier in the vicinity B_1^δ, when the strategy shown in Fig. 7 prescribes the control $u = +1$. Here, the following inequality

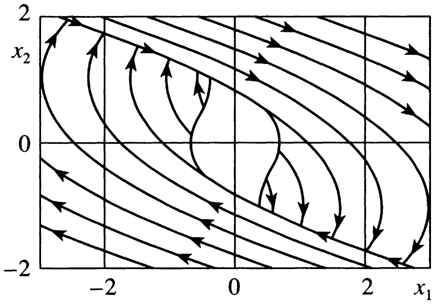

Fig. 9.

is satisfied

$$-\delta \le x_1^0 + (x_2^0+1)^2/2 - \bar{c} < 0 \tag{4.1}$$

Integration of the equations of motion in the segment [0, h] leads to the relations

$$x_2 = x_2^0 + \langle u\rangle h = x_2^0 + h$$

$$x_1 = x_1^0 + \langle x_2(t) + v\rangle h = x_1^0 + \langle x_2^0 + t + v\rangle h = x_1^0 + x_2^0 h + h^2/2 + \langle v\rangle h$$

where $\langle w\rangle = \frac{1}{h}\int_0^h w(t)dt$ is the time-average value of the quantity w. We will define, at a finite instant of time, the quantity f_h by the equation

$$f_h = x_1 + (x_2+1)^2/2 - \bar{c} = x_1^0 + x_2^0 h + \frac{h^2}{2} + \langle v\rangle h + (x_2^0 + h + 1)^2/2 - \bar{c} = \tilde{f}_h + \langle v\rangle h$$

$$\tilde{f}_h = (x_1^0 + (x_2^0+1)^2/2 - \bar{c}) + h^2 + (2x_2^0+1)h$$

We will require h to satisfy the bilateral inequality

$$-\delta \le f_* = \min_{\langle \bar{v}\rangle} f_h \le f_h \le \max_{\langle \bar{v}\rangle} f_h = f^* \le \delta \tag{4.2}$$

The left-hand inequality $f_* \ge -\delta$ remains true for any h, which follows from relation (4.1) and the positiveness of the coordinate x_2^0

$$f_* = \min_{\langle v\rangle} f_h = \tilde{f}_h - h \ge -\delta + (h^2 + 2x_2^0 h)$$

The right-hand inequality $f^* \le \delta$ yields

$$f^* = \max_{\langle v\rangle} f_h = \tilde{f}_h + h \le h^2 + 2(x_2^0+1)h \le \delta$$

From this, we obtain the following range of possible values of h

$$0 < h \le h_1, \quad h_1 = -(x_2^0+1) + \sqrt{(x_2^0+1)^2 + \delta}$$

Now let the point $x^0 = (x_1^0, x_2^0)$ be positioned above B_1, i.e., suppose the relation

$$0 \le x_1^0 + (x_2^0+1)^2/2 - \bar{c} \le \delta \tag{4.3}$$

is satisfied, and the synthesis in Fig. 7 prescribes player P the control $u = -1$.

For the motion in time interval [0, h] we obtain the relations

$$x_2 = x_2^0 + \langle u\rangle h = x_2^0 - h$$

$$x_1 = x_1^0 + \langle x_2(t) + v\rangle h = x_1^0 + \langle x_2^0 - t + v\rangle h = x_1^0 + x_2^0 h - h^2/2 + \langle v\rangle h$$

We will define the quantity f_h by the equation

$$f_h = x_1 + (x_2+1)^2/2 - \bar{c} = \hat{f}_h - h + \langle v\rangle h, \quad \hat{f}_h = x_1^0 + (x_2^0+1)^2/2 - \bar{c}$$

We will check that the chain of inequalities (4.2) is satisfied. The right-hand inequality ($f^* \le \delta$) is true by virtue of the positiveness of the coordinate x_2^0 for any selection of h

$$f^* = \max_{\langle v\rangle} f_h = \hat{f}_h \le \delta$$

Further, we will determine the range of variation of h for which the left-hand inequality $f_* \geq -\delta$ is satisfied. We have

$$f_* = \min_{\langle v \rangle} f_h = \hat{f}_h - 2h \geq -2h$$

From this we obtain

$$0 < h \leq h_2 = \delta/2$$

Thus, the value of step h that is suitable for the entire vicinity B_1^δ satisfies the inequality $0 < h \leq \bar{h}$, where

$$\bar{h} = \min[h_1, h_2]$$

It can be verified that the quantity $\bar{h}$ decreases monotonically as x_2^0 increases, and therefore the selection of step h from the initial data is suitable for further motion along barrier B_1, when the coordinate x_2 decreases. Such a choice of the step restrains constructive motion within the vicinity B_1^δ under any actions of player E. We will show that in this case the phase point will reach the δ-vicinity of region G in a finite time.

The limiting constructive motion (sliding regime) as $\delta \to 0$ along barrier B_1 corresponds to the control v of player E and to a certain equivalent control $\bar{u}$ of player P. Here, the minimum absolute value of the velocity of motion along the barrier

$$\min_{v} \sqrt{\bar{u}^2 + (x_2 + v)^2}$$

is achieved when $v = -1$. To calculate the worst time of motion towards region G for player P, we substitute the indicated controls into the equations of motion and write the condition for the limiting motion (sliding regime) to lie on the barrier

$$\dot{x}_1 = x_2 - 1, \quad \dot{x}_2 = \bar{u}, \quad x_1 + (x_2 + 1)^2/2 = \bar{c}$$

Differentiating the last equation with respect to time, we obtain an expression for an effective control

$$\dot{x}_2 = \bar{u} = -\frac{x_2 - 1}{x_2 + 1}$$

that satisfies the condition $|\bar{u}| \leq 1$.

The time of motion from the initial point $x^0 = (x_1^0, x_2^0)$ to a certain different point $x = (x_1, x_2)$ of barrier B_1 is written in the form

$$T = \int_{x_2}^{x_2^0} \frac{y+1}{y-1} dy = x_2^0 - x_2 + 2\ln\frac{x_2^0 - 1}{x_2 - 1}$$

It is equal to infinity if the finite point coincides with the point C (Fig. 3). The time of motion to the δ-vicinity of region G, when $x_2 - 1 < \delta$, has the order $T = \text{const} - \ln \delta^2$. The time of motion to point C may be finite if player E uses a control other than $v = -1$.

5. Guaranteeing strategies in the region G

The strategies of players P and E, presented in Figs. 7 and 8 respectively, belong to the entire game space. The values of the controls are indicated by their extreme values ± 1 or by letters U, V. This means that any value of the control that does not exceed unity in modulus is considered to be optimum. For simplicity, the symbols U and V can be given one of the values ± 1, in which case the strategies will be piecewise-constant functions with one continuous line dividing the values +1 and -1. For player P (Fig. 7), the dividing line comprises barriers and hyperbolae (2.1). For player E (Fig. 8) this line is composed of barrier B_1 and the v-dispersal line.

The discussions in the previous section enable us to give the following substantiation of players' strategies guaranteeing them the game value (3.2). For player P, substantiation is required for the regions X_0 and G_2. Region X_0 was examined above. In region G_2 the substantiation is based on the secondary game of bringing to hyperbolae (2.1). In region G_1, the behaviour of player P is arbitrary (Fig. 7), and he/she must not worry about guaranteeing the initial value $S(x_0)$.

For player E, substantiation of the guarantee is required only for region G. Although the behaviours of the players in regions G_1 and G_2 differ qualitatively, it is convenient to give a substantiation for region G as a whole. In region G, player E can, using the strategy shown in Fig. 8, ensure non-negativity of the total time derivative of the function $Q(x)$ – the game value in this region. In subregion G_1, the derivative is positive owing to the property of the quality function $S(x)$. In subregion G_2, this derivative is non-negative or equal to zero (for the optimum response of the first player), as follows from the construction of the solution of the secondary game of bringing to hyperbolae (2.1). On the v-dispersal line, the derivative of the non-smooth function $Q(x)$ according to known results[10] has the form (for example, when $x_2 > 0$)

$$\dot{Q}_2(x_1, x_2) = \max[\dot{V}_1(x_1, x_2), \dot{V}_2(x_1, x_2)]$$

By selecting any of the controls ± 1, player E will obtain non-negativity of one of the quantities $\dot{V}_1$ and $\dot{V}_2$, which will give $\dot{Q} \geq 0$.

We emphasize that the players' optimum strategies are extremely non-unique in game (1.1), (1.2), and the given strategies are only some of those possible. In particular, for more flexible behaviour of the players, it is possible to use strategies of the form $u = u(x, s)$, where the scalar s is equal to the minimum value of the quality function S that is observed from the instant of the start of motion (in the past). It is obvious that s does not exceed the current value of the quality function, $s \leq S(x)$. The controller must keep in memory and renew the value of the current minimum s. For example, with motion in the primary region G_1, player E, knowing that $s < S(x)$, can behave passively until the surface $s = S(x)$ or some vicinity of it is reached.

6. Construction of the solution for a quality function of general form

Let us consider the extension of the problem to the case where the objective function is the quadratic form $S^2(x) = \langle x, Ax\rangle$, $x = (x_1, x_2)$, with a symmetric positive definite matrix A.

The equality $S(x_1, x_2) = \rho$ on the plane specifies a family of ellipses with centre at the origin of coordinates. It is convenient to specify the parameterization of the family by the value of the eccentricity ε and the slope of the major axis α. Then the expression for the function S is written in the form of function (1.3). Note that the solution of the game problem for the case of quality function (1.3) does not present any essential difficulties. All conclusions drawn earlier for the case of circles remain true.

From the solution given earlier for zero eccentricity it can be seen that, to find the limit value of ρ_0, it is sufficient to find on the level $x_2 = 1$ a point C such that the branches of the parabolae $x_1 + (x_2 \pm 1)^2/2 = C_0$ departing from it will at the same time contact a certain representative of the family of ellipses. The ellipse found in this way will give the required value of ρ_0 (Fig. 10).

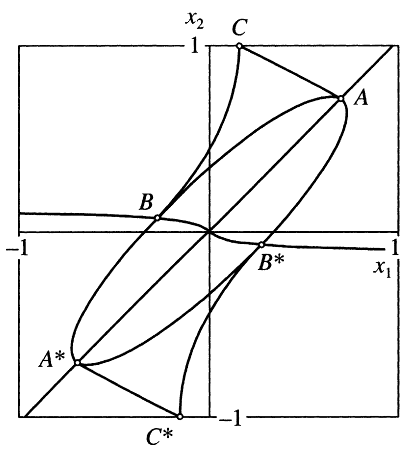

Fig. 10.

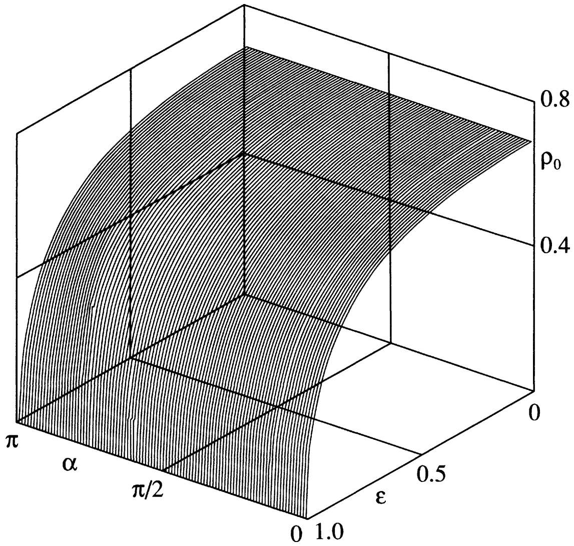

Fig. 11.

The quantity ρ_0 can be found in the following way. Let the points of contact of the parabolae and the ellipse be specified by the coordinates (x, y) and $(\tilde{x}, \tilde{y})$. The condition of contact formulated above is represented algebraically in the form of a system of five equations

$$\begin{aligned} &x + (y+1)^2/2 = \tilde{x} + (\tilde{y}-1)^2/2 + 2, \quad x^2 - \varepsilon^2 z^2(x, y) + y^2 = \rho_0^2 \\ &\tilde{x}^2 - \varepsilon^2 z^2(\tilde{x}, \tilde{y}) + \tilde{y}^2 = \rho_0^2 \\ &y - \varepsilon^2 z(x, y)\sin\alpha = (y+1)[x - \varepsilon^2 z(x, y)\cos\alpha] \\ &\tilde{y} - \varepsilon^2 z(\tilde{x}, \tilde{y})\sin\alpha = (\tilde{y}-1)[\tilde{x} - \varepsilon^2 z(\tilde{x}, \tilde{y})\cos\alpha] \end{aligned} \tag{6.1}$$

where the notation $z(x, y) = x\cos\alpha + y\sin\alpha$ is introduced. The first equation of system (6.1) specifies the condition that the parabolae run from the same point, and the following two equations specify the conditions that the points examined belong to one of the ellipses, while the conditions of contact of the parabolae are written in the form of the last two equations. The system was solved using the MAPLE software package. The graph of ρ_0 as a function of the parameters ε and α is presented in Fig. 11.

The constructions in the lower half-plane $x_2 < 0$ are carried out symmetrically.

The branches of the parabolae and ellipses form the limiting contour ∂G (Fig. 10). The game space, as in the case of circles, is divided into two subspaces: external to it and internal. The parameter ρ_0 serves as the game value in the external part. The interior of the contour, in turn, is also divided into two subspaces: a primary subregion, where the game value is specified by the original position of the players, and a secondary subregion where the first player has the chance to improve his/her position. This division is specified by the hyperbolae

$$x_2(x_1 - \varepsilon^2 z(x_1, y_1)\cos\alpha) + |x_1 - \varepsilon^2 z(x_1, y_1)\cos\alpha| - |x_2 - \varepsilon^2 z(x_1, y_1)\sin\alpha| = 0$$

Here it is possible to observe the existence of a v-dispersal curve, which is specified on the basis of geometrical considerations.

The solution procedure given above is suitable, generally speaking, for the case of an arbitrary quality function $S(x_1, x_2)$ that has a convex family of level lines.

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